GROUP THEORY 2024 - 25, SOLUTION SHEET 9

Exercise 1. To do yourself. Ask the assistant if something is unclear.

Exercise 2. Suppose that G admits a unique Sylow p-subgroup P. As gPg^{-1} is a subgroup of the same cardinality as P for all $g \in G$, we must have that $gPg^{-1} = P$ for all $g \in G$, i.e. P is normal in G. Conversely, suppose that P is a Sylow p-subgroup that is normal in G. Let Q be another Sylow p-subgroup. Then there exists $g \in G$ such that $Q = gPg^{-1} = P$, hence the desired unicity.

Exercise 3. As the order of any element of P is a power of p and the order of any element of Q is a power of q, we have that $P \bigcap Q = 1$. Moreover, P and Q are normal in G by the preceding exercise. By the 2nd isomorphism theorem, we have $PQ/Q \cong P/(P \bigcap Q)$ and thus $|PQ| = \frac{|P||Q|}{|P \bigcap Q|} = |P||Q| = |G|$. We thus obtain that $PQ = G$. We conclude by using the fact that if P, Q are normal in G, $PQ = G$ and $P \bigcap Q = 1$, then $G \cong P \times Q$.

Exercise 4. Observe first that $|A_5| = 60 = 2^2 \cdot 3 \cdot 5$. It follows from Sylow's third theorem that n_2, the number of Sylow 2-subgroups of A_5, satisfies $n_2 = [A_5, N_{A_5}(P)]$ for any Sylow 2-subgroup P. The order of Sylow 2-subgroups are the subgroups of order 4. We consider the Klein four-group $V_4 = \{1, (12)(34), (13)(24), (14)(23)\}$, seen in exercise 4, sheet 7, which satisfies

$$V_4 \triangleleft A_4 \leq A_5.$$

This a Sylow 2-subgroup of A_5. The permutations of this group all fix the element 5, and there are four other subgroups of this form, which fix the elements $1, 2, 3$ and 4 respectively. Hence there are at least five Sylow 2-subgroups of A_5. But because $V_4 \triangleleft A_4$ is normal, we have that $A_4 \leq N_{A_5}(V_4)$. It follows that $n_2 = 60/|N_{A_5}(V_4)| \leq 60/|A_4| = 60/12 = 5$. We found five different Sylow 2-subgroups of A_5, and we just argued that there cannot be more that five of them, which proves that we found them all.

Exercise 5. The exercise follows from the first Sylow theorem and the following lemma applied to a Sylow $p-$ subgroup:

Lemma 5.1: Let G be a group of order p^n for some $n > 0$ then G contains a normal subgroup of order p^k for all k such that $0 \leq k \leq n$.

Proof of Lemma: Recall that a group of order p^r has a non-trivial centre for all $r > 0$ (cf. Solution of Sheet 4 exercise 2.1). Using induction on n we obtain the lemma for $G/Z(G)$.

We can then conclude by induction and the correspondence theorem. We let the reader fill in the requisite details. □

Exercise 6. (1) Let P be any Sylow $p-$ subgroup of G and consider the action of H on the set of left co-sets G/P by left multiplication. Therefore we have the following equation:

$$|G/P| = \sum_{\text{Orbits } H\cdot x} \frac{|H|}{|\text{Stab}_H(x)|}.$$

Note that $|G/P|$ is an integer co-prime to p since P is a Sylow $p-$ subgroup. Since $p \mid |H|$ This implies that there exists $gP \in G/P$ with $\text{Stab}_H(gP) = H$. Therefore $hgP = gP$ for all $h \in H$ and hence $gHg^{-1} \subset P$. This shows that $H \subset gPg^{-1}$ which is a Sylow $p-$ subgroup of G.

(2) We show in the proof of the last part that given a Sylow $p-$ subgroup P and a subgroup H of order p^k, there exists $g \in G$ such that $gHg^{-1} \subseteq P$. If H is normal subgroup of G then this implies that $H \subseteq P$ for every Sylow $p-$ subgroup $P \subseteq G$.

Exercise 7. Let $g \in G$ then since K is a normal subgroup of G, we obtain that $gPg^{-1} \subseteq gKg^{-1} = K$. Since gPg^{-1} is also Sylow P subgroup of K we obtain by the second Sylow theorem that there exists $k \in K$ such that $kPk^{-1} = gPg^{-1}$. This implies that $(gk^{-1})P(gk^{-1})^{-1} = P$ and hence $gk^{-1} \in N_G(P)$. Therefore we obtain that $G = KN_G(P)$. □

Exercise 8. Denote by Q_n the wreath product $\mathbb{Z}/p\mathbb{Z} \wr \ldots \wr \mathbb{Z}/p\mathbb{Z}$, where the latter product contains n copies of $\mathbb{Z}/p\mathbb{Z}$. Firstly, we prove by induction that Q_n is a subgroup of S_{p^n} for all $n \geq 1$, i.e. there exists an injective group homomorphism $Q_n \hookrightarrow S_{p^n}$. For $n = 1$, this is clear, as the subgroup of S_p generated by a p-cycle is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Suppose the equality holds for all $1 \leq k < n$ and let us prove it for n. As $Q_n = Q_{n-1} \wr \mathbb{Z}/p\mathbb{Z}$ and we have injections $Q_{n-1} \hookrightarrow S_{p^{n-1}}$ and $\mathbb{Z}/p\mathbb{Z} \hookrightarrow S_p$, the cited injective homomorphism $S_n \wr S_m \to S_{nm}$ induces an injective homomorphism $Q_n = Q_{n-1} \wr \mathbb{Z}/p\mathbb{Z} \hookrightarrow S_{p^n}$, which proves the claim.

The exponent of p in the prime decomposition of $|S_{p^r}| = p^r!$ is $p^{\frac{p^r-1}{p-1}}$. Indeed, to find the desired value we have to count the number of multiples of p^k which are smaller than p^r for $0 < k \leq r$. For p^k this number is p^{r-k}, so their sum is:

$$\sum_{k=1}^{r} p^{r-k} = \sum_{k=0}^{r-1} p^k = \frac{p^r - 1}{p - 1}$$

Note: This is a particular case of a more general result, called Legendre's formula.
Let us now prove by induction that

$$|Q_n| = p^{\frac{p^n-1}{p-1}}$$

to conclude. For $n = 1$ the order is clearly equal to p. Suppose the equality holds for all $1 \leq k < n$ and let us prove it for n. In fact, it suffices to observe that, by definition:

$$|Q_n| = p|Q_{n-1}|^p$$

Hence, by the above said, Q_r is indeed a Sylow p-subgroup of S_{p^r}.

Exercise 9. We start by considering the prime decomposition $|G| = 48 = 2^4 \cdot 3$. By theorem 10 of the lecture notes, we know that the number n_2 of Sylow 2-subgroups must satisfy both $n_2 = 1 \pmod 2$, and $n_2 | 3$. Therefore we know that $n_2 \in \{1, 3\}$. If $n_2 = 1$, then the unique Sylow 2-subgroup P_2 is normal in G since Sylow 2-subgroups are conjugate to each other. If $n_2 = 3$, consider the action of G on the set of Sylow 2-subgroups of G. Since this is an action of G on a 3-elements set, this corresponds to a group homomorphism $\varphi : G \to S_3$. By the first isomorphism theorem we obtain that $G/\ker(\varphi) \cong im(\varphi)$. But since all Sylow 2-subgroups are conjugate to each other, φ is not the trivial map which means that $\ker(\varphi) \neq G$. Since the kernel cannot be trivial (because of size issues), it follows that $1 \neq \ker(\varphi) \triangleleft G$ is a non-trivial normal subgroup.